Multigrid Waveform Relaxation for Delay Partial Differential Equations

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ODEs and PDEs

- standard system of ODEs

$$\dot{v}(t) = f(t, v(t)), \qquad t \in [0, T]$$
$$v(0) = v_0$$

- often obtained by discretizing PDE
- typical example: heat equation

$$u_t = u_{xx} + f$$

- finite differences

$$\dot{u}_i = h^{-2}(u_{i-1} - 2u_i + u_{i+1}) + f_i$$

FDEs

- ordinary differential equation:
  solution depends on value at current time
- functional differential equation:
  solution can depend on whole history
- define function segment $v[t]$

$$v[t](s) = v(t+s), \qquad s \in [-\tau, 0]$$

- FDE

$$\begin{aligned} \dot{v}(t) &= f(t, v(t), v[t]), && t \in [0, T], \\ v(t) &= v_0(t), && t \in [-\tau, 0] \end{aligned}$$

DDEs and DPDEs

- specific subclass discussed here:
  delay differential equations with one constant delay

$$\begin{aligned} \dot{v}(t) &= f(t, v(t), v(t-\tau)), \qquad && t \in [0, T] \\ v(t) &= v_0(t), && t \in [-\tau, 0] \end{aligned}$$

- can come from discretizing delay partial differential equation
- heat equation with one constant delay

$$u_t = u_{xx} + u(t-\tau)$$

- finite differences

$$\dot{u}_i = h^{-2}(u_{i-1} - 2u_i + u_{i+1}) + u_i(t-\tau)$$

Example DPDEs

- Hutchinson equation with diffusion (population dynamics)

$$u_t = au_{xx} + bu(1 - K^{-1}u(t-\tau))$$

- distributed delay

$$u_t = au_{xx} + b(1 - K^{-1}\int_{-\infty}^{t} Q(t-x)u(s)ds)$$

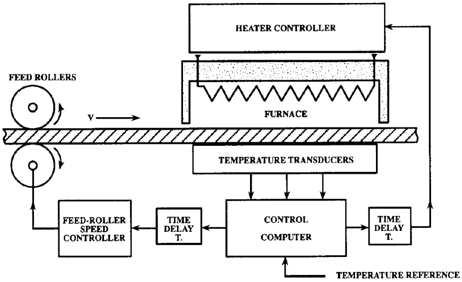
- other integrodifferential equations [Britton, 1990]

$$u_t = au_{xx} + (1 + bu - (1+b)g * u)$$

Examples (cont.)

- control theory

$$u_t = au_{xx} + v(g(u(t-\tau)))u_x + c[f(u(t-\tau)) - u]$$



- examples and picture from [Wu, 1991]
- see also [Kolmanovskii and Myshkis, 1999]

Model Problem

- 2D diffusion equation + term with constant delay

$$u_t = a(u_{xx} + u_{yy}) + bu(t-\tau)$$

- finite differences

$$\dot{u}_{i,j} = ah^{-2}\big(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\big) + bu_{i,j}(t-\tau)$$

- in matrix notation

$$\dot{v}(t) + Av(t) + v(t-\tau) = 0$$

A block-tridiagonal with tridiagonal blocks

Main Message

- good iterative methods exist for
  - systems of equations
  - discretizations of stationary PDEs
- waveform relaxation methods extend these to time dependent problems
  - systems of ODEs
  - discretizations of time-dependent PDEs
- here parabolic DPDEs
- WR allows sophisticated time stepping schemes: implicit Runge-Kutta, boundary value methods

Classical Iterative Methods

- system $Ax = b$
- splitting $A = M - N$
- iteration $Mx^{(\nu)} = Nx^{(\nu-1)} + b$
- M “simple”, but close to A
  - Jacobi: diagonal of A
  - Gauss-Seidel: lower triangular part of A
- iteration matrix $K = M^{-1}N$
- spectral radius

$$\rho(K) = \max\{|\lambda| : \lambda \in \sigma(K)\}$$

- $\rho < 1 \Rightarrow$ convergence, the smaller the better

Waveform Relaxation for ODEs

- iterative method for system of ODEs

$$\dot{v} + Av = 0$$

- splitting $A = M - N$

$$\dot{v}^{(\nu)} + Mv^{(\nu)} = Nv^{(\nu-1)}$$

- e.g. Jacobi

$$\dot{v}_i^{(\nu)} + a_{ii} v_i^{(\nu)} = -\sum_{j \neq i} a_{ij} v_j^{(\nu-1)}$$

Waveform Relaxation for DDEs

- DDE

$$\dot{v} + Av + v(t-\tau) = 0$$

- same splitting $A = M - N$
- delay term from previous iteration: Picard WR

$$\dot{v}^{(\nu)} + Mv^{(\nu)} = Nv^{(\nu-1)} - v^{(\nu-1)}(t-\tau)$$

- delay term from current iteration: non-Picard WR

$$\dot{v}^{(\nu)} + Mv^{(\nu)} + v^{(\nu)}(t-\tau) = Nv^{(\nu-1)}$$

Jacobi WR for DDEs

- Jacobi Picard

$$\dot{v}_i^{(\nu)} + a_{ii} v_i^{(\nu)} = -\sum_{j \neq i} a_{ij} v_j^{(\nu-1)} - v_j^{(\nu-1)}(t-\tau)$$

- sequence of scalar ODEs
- Jacobi non-Picard

$$\dot{v}_i^{(\nu)} + a_{ii} v_i^{(\nu)} + v_j^{(\nu)}(t-\tau) = -\sum_{j \neq i} a_{ij} v_j^{(\nu-1)}$$

- sequence of scalar DDEs
- similar for Gauss-Seidel

Convergence Analysis

- error bounds for Picard WR applied to general non-linear FDEs [Zubik-Kowal and Vandewalle, 1999]
- here: extend more quantitative analysis for linear ODEs [Miekkala and Nevanlinna, 1987]
- error iteration (Picard WR)

$$\dot{e}^{(\nu)}(t) + Me^{(\nu)}(t) = Ne^{(\nu-1)}(t) - e^{(\nu-1)}(t-\tau)$$

- iteration operator $\mathcal{K}$

$$e^{(\nu)} = \mathcal{K}e^{(\nu-1)}$$

- convergence determined by spectrum of $\mathcal{K}$
- consider $e^{(\nu)} \in L^p(0,\infty)$

Fourier-Laplace Analysis

- Laplace transform error iteration

$$\tilde{e}^{(\nu)}(z) = K(z)\tilde{e}^{(\nu-1)}(z)$$

- Picard WR symbol

$$K(z) = (zI + M)^{-1}(-e^{-\tau z}I + N)$$

- spectral radius of the iteration operator $\mathcal{K}$ in $L_p(0,\infty)$

$$\rho(\mathcal{K}) = \sup_{\Re z \geq 0} \rho(K(z)) = \sup_{\xi \in \mathbf{R}} \rho(K(i\xi))$$

- each $\rho(K(z))$ by standard Fourier analysis

Multigrid Waveform Relaxation

- convergence of Jacobi and Gauss-Seidel methods depends on h
- very slow unless grids are very coarse
- for elliptic PDEs: multigrid
- idea: use calculations on a coarse grid (cheaper) to accelerate iteration on fine grid
  - given approximation y to solution of $Ax = b$
  - write $x = y + e$
  - correction e is solution of $Ae = b - Ay = d$
  - solve this on coarse grid
- can be extended to time dependent PDEs/DPDEs

Two-Grid Iteration

- pre-smooth with Picard WR

$$\dot{v}_h^{(1)} + M_h v_h^{(1)} = N_h v_h^{(0)} - v_h^{(0)}(t-\tau)$$

- coarse grid correction
  - calculate defect

$$d_h = \dot{v}_h^{(1)} + A_h v_h^{(1)} + v_h^{(1)}(t-\tau)$$

  - transfer defect to coarse grid $d_H = Rd_h$
  - solve coarse-grid equivalent of defect equation

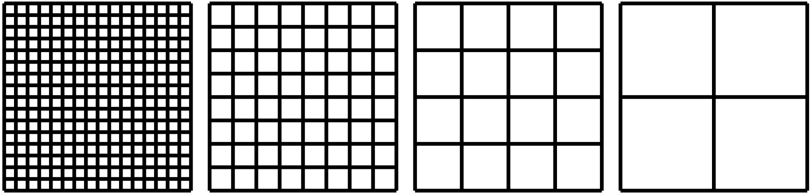
$$\dot{v}_H + A_H v_H - v_H(t-\tau) = d_H$$

  - transfer correction to fine grid $e_h = Pv_H$
  - correct current approximation $v_h^{(2)} = v_h^{(1)} - e_h$
- post-smooth with Picard WR

$$\dot{v}_h^{(3)} + M_h v_h^{(3)} = N_h v_h^{(2)} - v_h^{(2)}(t-\tau)$$

Multigrid Iteration

- Picard WR can be replaced by non-Picard WR
- ν_1 pre- and ν_2 post-smoothing steps can be applied
- defect equation has exactly the same form as original equation
- idea can be applied recursively: multigrid



Convergence Analysis

- two-grid WR symbol by Laplace transforming error iteration

$$
\begin{aligned}
M(z) &= K^{\nu_2}(z)C(z)K^{\nu_1}(z), \\
C(z) &= I - PL_H(z)^{-1}RL_h(z), \\
L_H(z) &= (z + e^{-\tau z})I + A_H, \\
L_h(z) &= (z + e^{-\tau z})I + A_h, \\
K(z) &= (zI + M_h)^{-1}(-e^{-\tau z}I + N_h)
\end{aligned}
$$

- spectral radius of the iteration operator $\mathcal{M}$ in $L_p(0,\infty)$

$$
\rho(\mathcal{M}) = \sup_{\Re z \geq 0} \rho(M(z)) = \sup_{\xi \in \mathbf{R}} \rho(M(i\xi))
$$

- each $\rho(M(z))$ by standard two-grid Fourier analysis

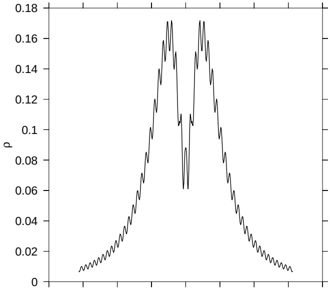
Results for Constant Coefficients (1/3)

$$u_t(t) = a\big(u_{xx}(t) + u_{yy}(t)\big) + bu(t-\tau)$$

- $L = 10,\ a = 1,\ b = -1,\ \tau = 1,\ M = 32$
- Picard or non-Picard red-black Gauss-Seidel WR smoother
- $\nu_1 = \nu_2 = 1$
- full weighting restriction, bilinear interpolation
- two-grid Fourier-Laplace analysis
- numerical results
  - multigrid V-cycle, 5 levels
  - scalar ODEs/DDEs solved with BDF2, $\Delta t = 0.1$
  - 40 iterations
  - geometric average of $\|e^{(\nu)}\|/\|e^{(\nu-1)}\|$ for last 20 iterations

Results for Constant Coefficients (2/3)

$\rho(M(i\xi))$ by two-grid analysis



- same graph for all a, b, L and τ such that $a\tau L^{-2} = 10^{-2}$ and $b\tau = -1$
- delay manifests itself as a wiggle on top of the curve for the equation without delay ($b = 0$)
- amplitude and frequency depend on the choice of parameters

Results for Constant Coefficients (3/3)

spectral radius (two-grid analysis)

τ	L	Picard	non-Picard
1	1	0.1625	0.1624
1	2	0.1626	0.1620
1	5	0.1651	0.1630
2	1	0.1623	0.1625
2	2	0.1627	0.1625
2	5	0.1637	0.1650

numerical convergence rates (multigrid)

τ	L	Picard	non-Picard
1	1	0.1374	0.1161
1	2	0.1951	0.1477
1	5	0.2037	0.1797
2	1	0.1372	0.1168
2	2	0.2021	0.1124
2	5	0.1157	0.1089

- results agree quite well (despite many approximations)
- for large L numerical convergence becomes more erratic, but the methods are still efficient

Results for Varying Coefficients (1/2)

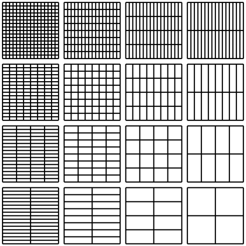
- ideas are more generally applicable
- diffusion equation with varying coefficients and constant delay
  $u_t = (au_x)_x + (bu_y)_y + cu + du(t-\tau) + f$

$$a(x,y,t) = \exp(10(x-y)\sin(t)), \qquad c(x,y,t) = 2 - \exp(-t),$$
$$b(x,y,t) = \exp(-10(x-y)\cos(\pi t)), \quad d(x,y,t) = 1 + \exp(t)$$

- $f(x,y,t)$ such that $u(x,y,t) = x + y + t$
- anisotropic problem:
  a and b depend strongly on direction of diffusion
- standard multigrid methods fail

Results for Varying Coefficients (2/2)

- use "multigrid as smoother" (MGS)
- same simple smoothers as before, but extended hierarchy of coarse grids
- 10 iterations
- average convergence factor over last 5 0.0557 (Picard and non-Picard smoother)
- spatial discretization error in 4 iterations (no discretization error in time)



Concluding Remarks

- quantitative convergence estimates for semi-discretized DPDE
- roughly speaking same as for PDE
- mesh-size independent convergence through multigrid
- simple model problem, but methods are easy to extend
- future research:
  - influence of time discretization
  - treatment of non-linearities
  - variable and state-dependent delays
  - cfr. previous talk by Nicola Guglielmi