Functional Calculus for the Spectral Analysis of Iterative Methods

Jan Van lent

Katholieke Universteit Leuven
Department of Computer Science

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BICS, University of Bath

Iterative Methods for Time-Dependent PDEs

- model problem

$$u_t = u_{xx} + u_{yy} + f$$

- discretise space and time

$$\frac{u_i - u_{i-1}}{\Delta t} = L u_i + f_i$$

- iterate ($L = L^+ + L^-$)

$$\frac{u_i^{(\nu)} - u_{i-1}^{(\nu)}}{\Delta t} = L^+ u_i^{(\nu)} + L^- u_i^{(\nu-1)} + f_i$$

Convergence Analysis

- iteration

$$u^{(\nu)} = \mathcal{K} u^{(\nu-1)} + \varphi$$

- spectral radius $\rho(\mathcal{K})$
- convergence if $\rho < 1$, the smaller the better
- for iterative methods for ODEs

$$\rho(\mathcal{K}) = \max_{z \in \Sigma} \rho(K(z))$$

- matrix-valued function

$$K : \mathbb{C} \to \mathbb{C}^{m \times m} : z \to K(z) = (zI - L^+)^{-1} L^-$$

- complex values

$$\Sigma \subset \mathbb{C} \cup \{\infty\}$$

Convergence Analyses

- many different methods
  - time stepping, continuous/discrete waveform relaxation
  - initial value problems on finite/infinite intervals, periodic problems
  - different iterative methods: Jacobi/Gauss-Seidel, multigrid
- convergence analyses similar
- but different enough to be annoying
- different tools
  - linear algebra
  - Volterra convolution operators
  - Laplace transforms
  - discrete Volterra convolution operators
  - discrete Laplace transforms (generating functions)
  - periodic convolution operators
- convinced that it could fit into one theory

An Interesting Theorem . . .

- matrix $T \in \mathbb{C}^{m \times m}$
- function $f : \mathbb{C} \to \mathbb{C}$
- matrix $f(T) \in \mathbb{C}^{m \times m}$
- spectral mapping theorem

$$\sigma(f(T)) = f(\sigma(T))$$

- suppose $f(z) = \rho(F(z))$
- would give $\rho(F(T)) = \rho(F(\sigma(T)))$
- $F = K$, $\Sigma = \sigma(T)$

. . . But not Quite What We Need

- missing maximum
- function $f(z) = \rho(F(z))$ is not analytic
- but on the right track
- solution: extend from scalar functions

$$f : \mathbb{C} \to \mathbb{C}$$

to matrix-valued functions

$$F : \mathbb{C} \to \mathbb{C}^{m \times m}$$

Functional Calculus for Scalar Functions

- function of a matrix
- well known examples: T^2, $\exp(T)$, $\cos(T)$, etc.
- many (equivalent) definitions
- polynomial or series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots$$

- definition 1

$$f(T) = c_0 I + c_1 T + c_2 T^2 + \ldots$$

Calculation Rules

- linearity

$$\alpha f(T) + \beta g(T) = (\alpha f + \beta g)(T)$$

- multiplication

$$f(T) \cdot g(T) = (f \cdot g)(T)$$

- function composition

$$g(f(T)) = (g \circ f)(T)$$

Spectral Mapping Theorem

- eigenvalue decomposition

$$T = V \operatorname{diag} \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} V^{-1}$$

- definition 2

$$f(T) = V \operatorname{diag} \begin{bmatrix} f(\lambda_1) & \cdots & f(\lambda_n) \end{bmatrix} V^{-1}$$

- spectral mapping theorem

$$\sigma(f(T)) = \{f(z) : z \in \sigma(T)\} =: f(\sigma(T))$$

- for general matrices Jordan or Schur

Complex Analysis

- Cauchy's integral formula

$$f(t) = \oint \frac{f(z)}{z-t} d\bar{z}$$

- definition 3

$$f(T) = \oint f(z)(zI - T)^{-1} d\bar{z}$$

- f holomorphic in a neighborhood of $\sigma(T)$
- works for bounded linear operators
- can be extended to closed linear operator
- $\sigma(T)$ need not be connected

Functional Calculus for Matrix-Valued Functions

- matrix $T \in C^{n \times n}$
- matrix-valued function

$$F : \mathbb{C} \to \mathbb{C}^{m \times p}$$

- F holomorphic in a neighborhood of $\sigma(T)$
- definition $F(T) \in \mathbb{C}^{mn \times pn}$

$$F(T) = \oint F(z) \otimes (zI - T)^{-1} d\bar{z}$$

- Kronecker or tensor product $A \otimes B = [a_{ij} B]$

Calculation Rules

- linearity

$$\alpha F(T) + \beta G(T) = (\alpha F + \beta G)(T)$$

- multiplication

$$F(T) \cdot G(T) = (F \cdot G)(T)$$

- function composition

$$G(F(T)) = (G \circ F)(T)$$

The Main Theorem

- spectral mapping theorem

$$\sigma(F(T)) = \bigcup_{z \in \sigma(T)} \sigma(F(z)) =: \sigma(F(\sigma(T)))$$

- corollary

$$\rho(F(T)) = \max_{z \in \sigma(T)} \rho(F(z))$$

- valid for
  - any bounded operator T
  - any matrix-valued function F
    holomorphic in a neighbourhood of $\sigma(T)$

The Main Theorem

$\sigma_\infty(T) = \sigma(T) \cup \{\infty\}$

- spectral mapping theorem

$$\sigma(F(T)) = \bigcup_{z \in \sigma_\infty(T)} \sigma(F(z)) =: \sigma(F(\sigma_\infty(T)))$$

- corollary

$$\rho(F(T)) = \max_{z \in \sigma_\infty(T)} \rho(F(z))$$

- valid for
  - any closed operator T
  - any matrix-valued function F holomorphic in a neighbourhood of $\sigma(T)$ and at ∞

Iterative Methods for ODEs

- equation

$$(I \otimes T)u = (L \otimes I)u + f$$

- iteration $(L = L^+ + L^-)$

$$(I \otimes T)u^{(\nu)} = (L^+ \otimes I)u^{(\nu)} + (L^- \otimes I)u^{(\nu-1)} + f$$

- iteration operator

$$F(T) = (I \otimes T - L^+ \otimes I)^{-1}(L^- \otimes I)$$

- matrix valued function

$$F(z) = (zI - L^+)^{-1}L^-$$

- analytic in $\mathbb{C} \setminus \sigma(L^+)$ (and at infinity)
- assume $\sigma(L^+) \cap \sigma(T) = \phi$

Continuous Waveform Relaxation

- equation

$$\dot{u} = Lu + f$$

- together with initial condition

$$u(0) = u_0$$

- or periodicity

$$u(0) = u^{(\nu)}(t_F)$$

- iteration

$$\dot{u}^{(\nu)} = L^+ u^{(\nu)} + L^- u^{(\nu-1)} + f$$

Finite Time Intervals

- Banach space $X = C[0, t_F]$
- operator $(Tx)(t) = \dot{x}(t)$, with domain

$$\mathcal{D}(T) = \{x : \dot{x} \in C[0, t_F], x(0) = 0\}$$

- closed unbounded operator with $\sigma(T) = \phi$
- extended spectrum $\sigma_\infty(T) = \{\infty\}$
- result $\rho(F(T)) = \rho(F(\infty)) = 0$

Infinite Time Intervals

- Banach space $X = L^p([0, \infty], \mathbb{C})$
- operator $(Tx)(t) = \dot{x}(t)$, with domain

$$\mathcal{D}(T) = \{x : x \text{ is absolutely continuous on } [0, a] \text{ for any } a > 0, \\ \dot{x} \in X, x(0) = 0\}$$

- closed unbounded operator with $\sigma(T) = \mathbb{C}^-$
- result $\rho(F(T)) = \sup_{z \in \mathbb{C}^-} \rho(F(z))$

Periodic Problems

- Banach space $X = C[0,1]$
- operator $(Tx)(t) = \dot{x}(t)$, with domain

$$\mathcal{D}(T) = \{x : \dot{x} \in C[0,1],\, x(0) = x(1)\}$$

- closed unbounded operator with $\sigma(T) = 2\pi i\mathbb{Z}$
- result $\rho(F(T)) = \sup_{j\in\mathbb{Z}} \rho(F(2\pi ij))$

Discrete Waveform Relaxation

- T itself by functional calculus
- implicit Euler $(Tu)_i = \frac{u_i - u_{i-1}}{\Delta t}$
- shift operator $(Sx)_i = x_{i-1}$
- function $g(z) = \frac{1-w}{\Delta t}$
- operator $T = g(S)$

Finite Sequences

- Banach space $X = l^p(n, \mathbb{C})$
- matrix

$$S = \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

- Toeplitz matrix with $\sigma(S) = \{0\}$

Infinite Sequences

- Banach space $X = L^p(\infty, \mathbb{C})$
- operator $[x_1, x_2, \ldots] \rightarrow [0, x_1, x_2, \ldots]$

$$S = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \end{bmatrix}$$

- Toeplitz operator with $\sigma(S) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

Periodic Sequences

- Banach space $X = l^p(n, \mathbb{C})$
- matrix

$$S = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{bmatrix}$$

- circulant matrix with $\sigma(S) = \{\exp\left(\frac{2\pi i j}{n}\right), j = 0, \ldots, n-1\}$

Summary

$$\rho(F(T)) = \sup_{z \in \sigma(T)} \rho(F(z))$$

- continuous waveform relaxation
  - finite $\sigma(T) = \{\infty\}$
  - infinite $\sigma(T) = \mathbb{C}^-$ (or $i\mathbb{R}$)
  - periodic $\sigma(T) = 2\pi i \mathbb{Z}$
- discrete waveform relaxation $\sigma(T) = \sigma(G(S)) = \sigma(G(\sigma(S)))$
  - finite $\sigma(S) = \{0\}$
  - infinite $\sigma(S) = \{w \in \mathbb{C} : |w| \leq 1\}$ (or $|w| = 1$)
  - periodic $\sigma(S) = \{w = \exp\left(\frac{2\pi ij}{n}\right), j = 0, \ldots, n-1\}$
- implicit Euler $G(w) = g(w) = \frac{1-w}{\Delta t}$

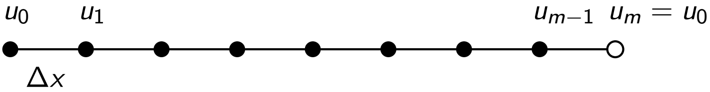
Multigrid Example

- elliptic partial differential equation : 1D Poisson

$$u_{xx} = f, \qquad u, f : [0,1] \to \mathbb{R}$$

- discretisation : finite differences

$$u_{i-1} - 2u_i + u_{i+1} = \Delta x^{-2} f_i$$



Iterative Solution

- system of equations

$$Lu = f, \qquad u, f \in \mathbb{R}^m$$

- classical iterations

$$L^+ u^{(\nu)} + L^- u^{(\nu-1)} = f$$

- e.g. Jacobi, **Gauss-Seidel**

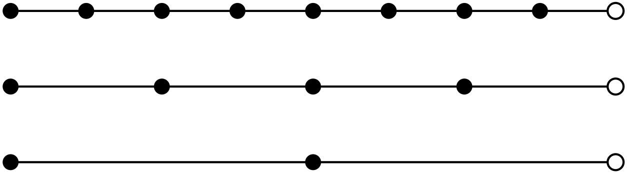
$$u_i^{(\nu)} = \frac{1}{2}(u_{i-1}^{(\nu)} + u_{i+1}^{(\nu-1)} - \Delta x^{-2} f_i)$$

Multigrid Principle

- convergence of Jacobi and Gauss-Seidel very slow
- for elliptic PDEs: multigrid
- idea: use calculations on a coarse grid (cheaper) to accelerate iteration on fine grid
  - given approximation v to solution of $Lu = f$
  - write $u = v + e$
  - correction e is solution of $Le = f - Lv = d$
  - solve this on coarse grid

Multigrid Components

- discretisations at different levels of coarseness



- smoothing : Gauss-Seidel
- restriction : $\bar{f}_i \leftarrow (d_{2i-1} + d_{2i} + d_{2i+1})/4$
- prolongation : $e_{2i} \leftarrow \bar{u}_i \, , \qquad e_{2i+1} \leftarrow (\bar{u}_i + \bar{u}_{i+1})/2$

Two-Grid Algorithm

$\text{mg}(\mathbf{u}^{(0)}, \mathbf{L}, \mathbf{f}) \rightarrow \mathbf{u}^{(3)}$

- $\mathbf{u}^{(1)} \leftarrow \text{smooth}(\mathbf{u}^{(0)}, \mathbf{L}, \mathbf{f}, \mu_1)$
- $\bar{\mathbf{f}} \leftarrow \mathbf{R}(\mathbf{f} - \mathbf{L}\mathbf{u}^{(1)})$
- $\bar{\mathbf{u}} \leftarrow \text{cgs}(\bar{\mathbf{L}}, \bar{\mathbf{f}})$
- $\mathbf{u}^{(2)} \leftarrow \mathbf{u}^{(1)} + \mathbf{P}\bar{\mathbf{u}}$
- $\mathbf{u}^{(3)} \leftarrow \text{smooth}(\mathbf{u}^{(2)}, \mathbf{L}, \mathbf{f}, \mu_2)$

$\text{cgs}(\bar{\mathbf{L}}, \bar{\mathbf{f}}) \rightarrow \bar{\mathbf{u}}$

- $\bar{\mathbf{u}} \leftarrow \bar{\mathbf{L}}^{-1}\bar{\mathbf{f}}$

$\text{smooth}(\mathbf{u}^{(0)}, \mathbf{L}, \mathbf{f}, \mu) \rightarrow \mathbf{u}^{(\mu)}$

- for $\nu = 1, \ldots, \mu$
  solve
  $\mathbf{L}^{+}\mathbf{u}^{(\nu)} = \mathbf{f} - \mathbf{L}^{-}\mathbf{u}^{(\nu-1)}$

Multigrid Algorithm

$\mathrm{mg}(\mathbf{u}^{(0)}, \mathbf{L}, \mathbf{f}) \rightarrow \mathbf{u}^{(3)}$

- $\mathbf{u}^{(1)} \leftarrow \mathrm{smooth}(\mathbf{u}^{(0)}, \mathbf{L}, \mathbf{f}, \mu_1)$
- $\bar{\mathbf{f}} \leftarrow \mathbf{R}(\mathbf{f} - \mathbf{L}\mathbf{u}^{(1)})$
- $\bar{\mathbf{u}} \leftarrow \mathrm{mg}(0, \bar{\mathbf{L}}, \bar{\mathbf{f}})$
- $\mathbf{u}^{(2)} \leftarrow \mathbf{u}^{(1)} + \mathbf{P}\bar{\mathbf{u}}$
- $\mathbf{u}^{(3)} \leftarrow \mathrm{smooth}(\mathbf{u}^{(2)}, \mathbf{L}, \mathbf{f}, \mu_2)$

$\mathrm{smooth}(\mathbf{u}^{(0)}, \mathbf{L}, \mathbf{f}, \mu) \rightarrow \mathbf{u}^{(\mu)}$

- for $\nu = 1, \ldots, \mu$
  solve
  $\mathbf{L}^{+}\mathbf{u}^{(\nu)} = \mathbf{f} - \mathbf{L}^{-}\mathbf{u}^{(\nu-1)}$

Two-Grid Operator

- multigrid iteration

$$u^{(\nu)} = Mu^{(\nu-1)} + \varphi$$

- two-grid

$$M = S^{\nu_2} C S^{\nu_1}$$
$$S = (L^+)^{-1} L^-$$
$$C = I - P(\overline{L})^{-1} R$$

Multigrid for the 1D Poisson Equation

$$u_{xx} = f$$

- coarse grid with $\bar{m}$ nodes

$$\begin{bmatrix} \bar{u}_0 & \bar{u}_1 & \cdots & \bar{u}_{\bar{m}-1} \end{bmatrix}$$

- fine grid with $m = 2\bar{m}$ nodes

$$\begin{bmatrix} u_0 & u_1 & u_2 & u_3 & \cdots & u_{m-2} & u_{m-1} \end{bmatrix}$$

- split into even and odd: $u_i^0 = u_{2i}$, $u_i^1 = u_{2i+1}$

$$\begin{bmatrix} u_0^0 & u_0^1 & u_1^0 & u_1^1 & \cdots & u_{\bar{m}-1}^0 & u_{\bar{m}-1}^1 \end{bmatrix}$$

- $\bar{u},\ u^0,\ u^1 \in \mathbb{R}^{\bar{m}}$

Differential Operator on the Coarse Grid

- grid spacing $\overline{\Delta x} = 2$
- operator $\overline{L} : \mathbb{R}^{\bar{m}} \rightarrow \mathbb{R}^{\bar{m}} : \bar{u} \rightarrow \bar{v} = \overline{L}\bar{u}$

$$\bar{v}_i = (\bar{u}_{i-1} - 2\bar{u}_i + \bar{u}_{i+1})/4$$

- factor 4 corresponds to $\overline{\Delta x}^2$
- stencil

$$\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} /4$$

- function $\overline{L} : \mathbb{C} \rightarrow \mathbb{C}$

$$\overline{L}(w) = (w - 2 + w^{-1})/4$$

- $\overline{L} = \overline{L}(S)$ with S the periodic shift operator

Differential Operator on the Fine Grid

- grid spacing $\Delta x = 1$
- operator $L : \mathbb{R}^{2\bar{m}} \to \mathbb{R}^{2\bar{m}} : u \to v = Lu$

$$v_i = u_{i-1} - 2u_i + u_{i+1}$$

- even points $\begin{bmatrix} \cdots & u_{i-1}^1 & \boxed{u_i^0} & u_i^1 & \cdots \end{bmatrix}$

$$v_i^0 = u_{i-1}^1 - 2u_i^0 + u_i^1$$

- odd points $\begin{bmatrix} \cdots & u_i^0 & \boxed{u_i^1} & u_{i+1}^0 & \cdots \end{bmatrix}$

$$v_i^1 = u_i^0 - 2u_i^1 + u_{i+1}^0$$

- matrix-valued function $L : \mathbb{C} \to \mathbb{C}^{2\times 2}$

$$L(w) = \begin{bmatrix} -2 & w+1 \\ 1+w^{-1} & -2 \end{bmatrix}$$

Full Weighting Restriction

- operator $R : \mathbb{R}^{2\bar{m}} \to \mathbb{R}^{\bar{m}} : u \to \bar{v} = Ru$

$$\bar{v}_i = (u_{2i-1} + 2u_{2i} + u_{2i+1})/4$$

- stencil

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} /4$$

- even-odd

$$\bar{v}_i = (u^1_{i-1} + 2u^0_i + u^1_i)/4$$

- matrix-valued function $R : \mathbb{C} \to \mathbb{C}^{2\times 1}$

$$R(w) = \begin{bmatrix} 2 & w+1 \end{bmatrix} /4$$

Bilinear Interpolation

- operator $P : \mathbb{R}^{\bar{m}} \to \mathbb{R}^{2\bar{m}} : \bar{u} \to v = P\bar{u}$

$$\begin{aligned} v_i^0 &= \bar{u}_i \\ v_i^1 &= (\bar{u}_i + \bar{u}_{i+1})/2 \end{aligned}$$

- stencil

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} /2$$

- matrix-valued function $P : \mathbb{C} \to \mathbb{C}^{1\times 2}$

$$P(w) = \begin{bmatrix} 1 \\ (1 + w^{-1})/2 \end{bmatrix}$$

Smoothing

- Jacobi

$$S_{\mathrm{J}} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & w+1 \\ 1+w^{-1} & 0 \end{bmatrix}$$

- red-black Gauss-Seidel $S_{\mathrm{RB}} = S_{\mathrm{B}} S_{\mathrm{R}}$

$$S_{\mathrm{R}} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & w+1 \\ 0 & 1 \end{bmatrix}$$

$$S_{\mathrm{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 1+w^{-1} & 0 \end{bmatrix}$$

Multigrid for the 2D Poisson Equation

$$u_{xx} + u_{yy} = f$$

- differential operator on the fine grid $\Delta x = \Delta y = 1$
- stencil

$$\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$$

- split fine grid in even-even, even-odd, odd-even, odd-odd
- matrix-valued function $L : \mathbb{C}^2 \to \mathbb{C}^{4\times 4}$

$$\begin{aligned} L(w_x, w_y) &= L(w_x) \otimes I_2 + I_2 \otimes L(w_y) \\ &= \begin{bmatrix} -4 & w_y + 1 & w_x + 1 & 0 \\ 1 + w_y^{-1} & -4 & 0 & w_x + 1 \\ 1 + w_x^{-1} & 0 & -4 & w_y + 1 \\ 0 & 1 + w_x^{-1} & 1 + w_y^{-1} & -4 \end{bmatrix} \end{aligned}$$

Multigrid for the 2D Heat Equation

$$u_t = u_{xx} + u_{yy} + f$$

- combine time-dependent and multigrid
- matrix-valued function $M : \mathbb{C}^3 \rightarrow \mathbb{C}^{4\times 4}$

$$M(w_t, w_x, w_y) =$$
$$\begin{bmatrix} \times&\times&\times&\times \\ \times&\times&\times&\times \\ \times&\times&\times&\times \\ \times&\times&\times&\times \end{bmatrix}^{\nu_2} \left(\begin{bmatrix} \times&&& \\ &\times&& \\ &&\times& \\ &&&\times \end{bmatrix} - \begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix} [\times]^{-1} [\times\times\times\times] \right) \begin{bmatrix} \times&\times&\times&\times \\ \times&\times&\times&\times \\ \times&\times&\times&\times \\ \times&\times&\times&\times \end{bmatrix}^{\nu_1}$$

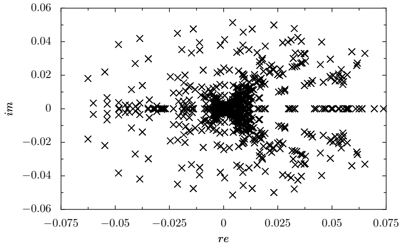
- convergence for infinite grids ($\mathbb{S} = \{w \in \mathbb{C} : |w| = 1\}$)

$$\rho(M) = \max_{(w_t, w_x, w_y) \in \mathbb{S}^3} \rho(M(w_t, w_x, w_y))$$

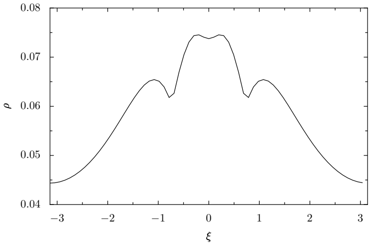
Pictures

- heat equation, two-grid, red-black Gauss-Seidel, full weighting restriction, bilinear interpolation, $\nu_1 = \nu_2 = 1$
- periodic grids $\bar{n}_x = \bar{n}_y = 16$
- implicit Euler $\Delta t = 1$
- $\rho = 0.074$

$\sigma(M(w_t, w_x, w_y))$
$n_t = 16$



$\max_{w_x, w_y} \rho(M(e^{i\xi}, w_x, w_y))$
$n_t = 64$



Ideas for the Future

- extend to differential algebraic equations (DAEs) could be straightforward

$$M\dot{u} = Lu + f, \qquad M \text{ singular}$$

- extend to delay differential equations (DDEs)

$$\dot{u}(t) = Lu(t) + u(t - \tau)$$

- multigrid for curl-curl
- operator-valued functions (Banach algebras and Gelfand theory)

$$T \in X, \qquad F : \mathbb{C} \to Y, \qquad F(T)$$

- continuation of largest eigenvalue

Delay Equations

$$\dot{u}(t) = Lu(t) + u(t - \tau)$$

- derivative operator

$$T_1 u(t) = \dot{u}(t)$$

- delay operator (continuous shift)

$$T_2 u(t) = u(t - \tau)$$

- we need
  - functional calculus for functions not analytic at infinity
    $T_2 = \exp(T_1)$ (formally)
    semigroup theory, Hille-Phillips functional calculus
  - functional calculus for commuting operators
    T_1 and T_2 commute
    joint spectrum
    $\sigma(F(T_1, T_2)) = \sigma(F(\sigma(T_1, T_2)))$

Further Reading

- WR for IVPs (Miekkala and Nevanlinna '87)
- WR for periodic ODEs (Vandewalle and Piessens '92, '93)
- WR for finite elements (Janssen and Vandewalle '96)
- functional calculus (Dunford and Schwartz; Taylor; Haase; . . .)
- MG (Briggs, Henson and McCormick; Trottenberg, Oosterlee and Schüller; Stüben)
- MG WR (Lubich and Ostermann '89; Vandewalle)
- Banach algebras (Rudin; Lax)
- tensor products (Reed and Simon '73; Ichinose '78)
- semigroup theory (Hille and Philips)
- joint spectra (Taylor '70; Vasilescu)